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PAPER

Robustness to Noise of Associative Memory Using Nonmonotonic Analogue Neurons

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SUMMARY In this paper, dependence of storage capacity of an analogue associative memory model using nonmonotonic neurons on static synaptic noise and static threshold noise is shown. This dependence is analytically calculated by means of the self-consistent signal-to-noise analysis (SCSNA) proposed by Shiino and Fukai. It is known that the storage capacity of an associative memory model can be improved markedly by replacing the usual sigmoid neurons with nonmonotonic ones, and the Hopfield model has theoretically been shown to be fairly robust against introducing the static synaptic noise. In this paper, it is shown that when the monotonicity of neuron is high, the storage capacity decreases rapidly according to an increase of the static synaptic noise. It is also shown that the reduction of the storage capacity is more sensitive to an increase in the static threshold noise than to the increase in the static synaptic noise.

key words: *nonmonotonic neuron, static synaptic noise, static noise in the threshold, storage capacity, SCSNA*

1. Introduction

The Hopfield-type associative memory model [1] has an advantage of having a simple structure but has a disadvantage of low storage capacity ($\alpha_C = 0.14$) [2]. Recently, however it has been shown that the storage capacity of an associative memory model can be improved markedly by replacing the usual sigmoid neurons with nonmonotonic ones [3]. Yoshizawa et al. [4] showed that the storage capacity of an associative memory model with optimal nonmonotonicity is $\alpha_C = 0.4$, which is approximately three times as large as that of the Hopfield model. On the other hand, the Hopfield model has theoretically been shown to be fairly robust against introducing static synaptic noise or nonlinearity of synapse [5]. In the Hopfield model, if synaptic weights are quantized into two levels (± 1), the storage capacity still remains at approximately $\alpha_C \approx 0.1$.

In this paper, a piecewise linear model of the nonmonotonic neuron is adopted [6], and the storage capacity of an associative memory model using the nonmonotonic neurons with two types of noise, that is, static synaptic noise and static threshold noise, is shown theoretically by means of the self-consistent signal-to-noise analysis (SCSNA) proposed by Shiino and Fukai [7], [8].

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2. Model

The two types of noise are considered, the static additional synaptic noise and the static noise in the threshold.

We begin by formulating a recurrent neural network with N analogue neurons to show how the SCSNA is applied to the network. The network dynamics are written in terms of internal potential u variables as

$$\frac{d}{dt}u_i = -u_i + \sum_{j \neq i}^{N} J_{ij}x_j + \delta_i,$$
$$x_i = F(u_i) \tag{1}$$
$$\delta_i \sim \mathcal{N}\left(0, \Delta_h^2\right), \tag{2}$$

where δ_i is the static noise in the threshold, which must have $O(1)$ in order to be signified as noise. The variable x_i denotes an output of the i-th neuron. The internal potential u variables are assumed to be connected with each other through the synaptic weights J_{ij} of a form

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu + \delta_{ij}, \qquad i \neq j, \tag{3}$$

where the first term in the right-hand side (rhs) corresponds to the correlation type learning. The static synaptic noise denoted by δ_{ij} in Eq. (3) is a random variable following the Gaussian distribution with mean 0 and variance Δ_J^2/N

$$\delta_{ij} \sim \mathcal{N}\left(0, \frac{\Delta_J^2}{N}\right), \qquad \delta_{ij} = \delta_{ji}. \tag{4}$$

In the present paper, the symmetric weight case, i.e., $J_{ij} = J_{ji}$, is considered. Note that the variance of noiseless synaptic weights of the first term in the rhs of Eq. (3) is $O(1/N)$. Therefore, the variance of the synaptic noise must be scaled by $1/N$ for the former and the latter to have the same order. Let us denote the p sets of random vectors to store by ξ_i^μ ($\mu = 1, 2, \cdots, p,\ i = 1, 2, \cdots, N$), where each element ξ_i^μ is an independent random variable taking a value of 1 or -1 with probability

$$\mathrm{P}[\xi^\mu = \pm 1] = \frac{1}{2}. \tag{5}$$

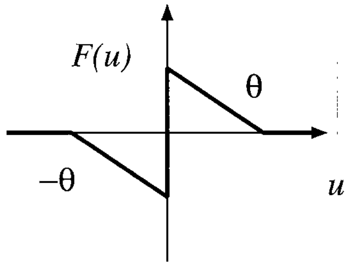

Fig. 1 Piecewise linear nonmonotonic output function.

Loading rate α is defined as ratio of the number of memory patterns p to the number of neurons N, that is, $p = \alpha N$.

The overlap between the μ-th memory pattern $\{\xi_i^\mu\}$ and the network state $\{x_i\}$ is now defined as

$$m_\mu = \frac{1}{N}\sum_{j=1}^{N} \xi_j^\mu x_j, \qquad \mu = 1, \cdots, \alpha N. \tag{6}$$

In the case that the output function $F(\cdot)$ is set to the sign function sgn$(\cdot)$, the overlap m_μ represents the direction cosine between the memory pattern $\{\xi_i^\mu\}$ and the network state $\{x_i\}$. The equilibrium of the dynamics (1) is obtained by setting $du_i/dt = 0$. The equilibrium in which the only first pattern is retrieved, i.e., $m_1 = O(1)$ and $m_\mu = O(1/\sqrt{N})$, where $\mu = 2, 3, \cdots, \alpha N$, shall be our focus. In this paper, we use the following odd function $F(\cdot)$ shown in Fig. 1 as an output function

$$F(u) = \begin{cases} -\dfrac{1}{\theta}u - 1, & -\theta < u < 0, \\ -\dfrac{1}{\theta}u + 1, & 0 < u < \theta, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

In this paper, we use $1/\theta$ as a parameter representing a degree of nonmonotonicity. In the case of $1/\theta \to 0$, the output function $F(\cdot)$ in Eq. (7) converges on the sign function sgn$(\cdot)$.

3. Results

3.1 SCSNA

The internal potential u_i of each neuron in the equilibrium state is represented by the weighted sum of outputs from the other neurons. The SCSNA has its basis in the systematic splitting of the internal potential into a signal part and a cross-talk noise part. Moreover the cross-talk noise part consists of two parts in the framework of the SCSNA. One is an effective self-coupling term, which comes from statistical correlations caused by the recurrent connections, the other obeys the Gaussian distribution with mean 0 and variance αr. The following results are obtained for any odd output function $F(\cdot)$

$$m = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) Y(z), \tag{8}$$

$$q = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) Y(z)^2, \tag{9}$$

$$U = \frac{1}{\sigma}\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) zY(z), \tag{10}$$

$$Y(z) = F\left(m + \Gamma Y(z) + \sigma z\right), \tag{11}$$

$$r = \frac{q}{(1-U)^2} \tag{12}$$

$$\Gamma = \frac{\alpha U}{1-U} + \Delta_J^2 U \tag{13}$$

$$\sigma = \sqrt{\alpha r + \Delta_J^2 q + \Delta_h^2}, \tag{14}$$

where $Y(z)$ is an effective output function of each neuron obtained by solving Eq. (11), m is the overlap for the target pattern $\{\xi_i^1\}$ defined in Eq. (6), q is the so-called Edwards-Anderson order parameter, U is a kind of the susceptibility, which measures sensitivity of neuron output with respect to the external input, r represents enhancement of the cross-talk noise caused by the recurrent connections, $\Gamma Y(z)$ denotes the effective self-coupling term, and σ^2 is the variance of the noise obeying the Gaussian distribution. According to the SCSNA, the macroscopic description of any microscopic state can be represented by the three order parameters of m, q, and U. The storage capacity can be calculated by solving Eqs. (8)–(14) self-consistently as follows. These equations are solved numerically. There is a transition at $\alpha = \alpha_C$. When the loading rate α is below α_C, Eqs. (8)–(14) have a nontrivial solution with $m \neq 0$. On the other hand, if α is above α_C, only the trivial solution with $m = 0$ exists. This critical value α_C is the storage capacity. Derivation of the above-mentioned order parameter equations from Eqs. (8) through (14) is given in Appendix.

3.2 Effects of Static Synaptic Noise

In this section, we set $\Delta_h = 0$. The dependence of storage capacity α_C on the static synaptic noise Δ_J and the nonmonotonicity $1/\theta$ is shown. Figure 2 illustrates the relationship of α_C against Δ_J for some fixed nonmonotonicity values $1/\theta = 0.0, 0.1, \cdots, 0.5$. As mentioned before, the conventional monotonic model corresponds to $1/\theta = 0$. The choice $1/\theta = 0.5$ gives the highest nonmonotonicity, since the storage capacity obtained by the SCSNA for the noiseless case $\Delta_J = 0$ tends to be larger than the actual values obtained by computer simulation for $1/\theta > 0.5$ [6]. Figure 3 illustrates the relationship of α_C against $1/\theta$ for some fixed static synaptic noise values $\Delta_J = 0.0, 0.1, \cdots, 0.6$. One may think that the storage capacity of the highest nonmono-

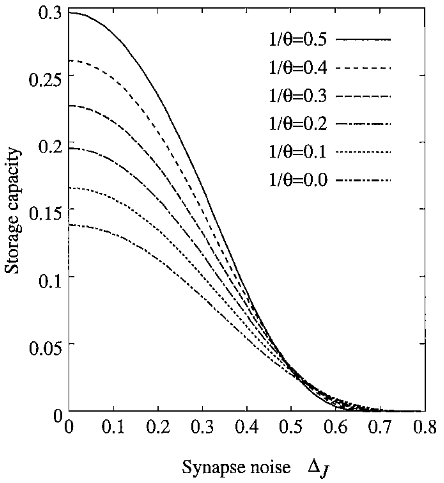

Fig. 2 The storage capacity α_C and static synaptic noise Δ_J for fixed $1/\theta$ ($\Delta_h = 0$).

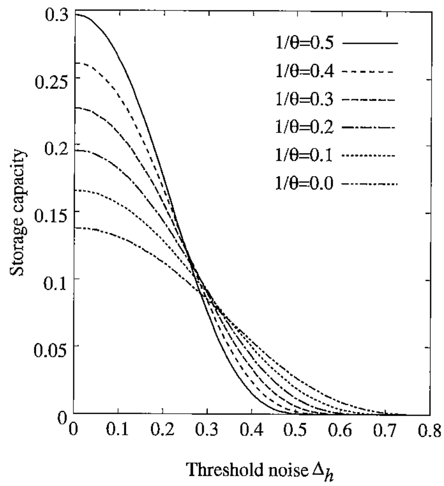

Fig. 4 The storage capacity α_C and threshold noise Δ_h for fixed $1/\theta$ ($\Delta_J = 0$).

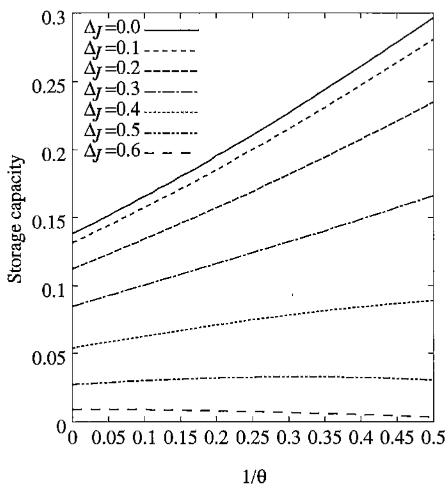

Fig. 3 The storage capacity α_C and nonmonotonicity $1/\theta$ for fixed Δ_J ($\Delta_h = 0$).

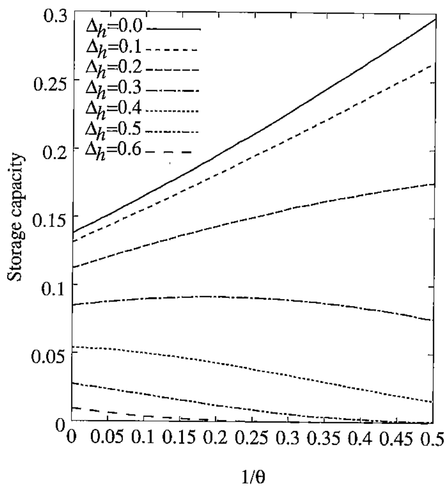

Fig. 5 The storage capacity α_C and nonmonotonicity $1/\theta$ for fixed Δ_h ($\Delta_J = 0$).

tonicity $1/\theta = 0.5$ does *not* coincide with the result of Yoshizawa et al. [4], where the storage capacity is $\alpha_C = 0.4$. The reason is difference of their output functions. A piecewise linear model discussed by Yoshizawa et al. can be analyzed by the SCSNA and the results coincide with each other [9], [10]. The following results shown in the present paper do *not* depend on detailed shape of output function qualitatively.

In the case of the Hopfield model, i.e., $1/\theta = 0$, the susceptibility U is always positive. In the case of the nonmonotonic model, however, U may be negative in the retrieval phase; therefore, the variance of effective noise σ^2 decreases more than in the Hopfield model. Hence, the storage capacity of the nonmonotonic model increases. In Fig. 2, the storage capacity vanishes at $\Delta_J \cong 0.8$ for all nonmonotonicity values. This is because, noting that r is proportional to q, αr in σ (Eq. (14)) can be neglected if Δ_J is large.

3.3 Effects of Static Noise in the Threshold

In this section, we set $\Delta_J = 0$. The dependence of storage capacity α_C on the static threshold noise Δ_h and the nonmonotonicity $1/\theta$ is shown. Figure 4 illustrates the relationship of α_C against Δ_h, for some fixed nonmonotonicity values $1/\theta = 0.0, 0.1, \cdots, 0.5$. Figure 5 illustrates the relationship of α_C against $1/\theta$, for some fixed noise in the threshold values $\Delta_h = 0.0, 0.1, \cdots, 0.6$.

The storage capacity is more sensitive to an increase in the threshold noise level than to an increase in the synaptic noise level. When the nonmonotonicity $1/\theta$ increases, the Edwards-Anderson order parameter q decreases. Therefore, the threshold noise is more effective than the synaptic noise, since Δ_h^2 is larger than $\Delta_J^2 q$ in the variance of the effective noise σ^2. These results regarding the threshold noise strongly depend on the dynamic range of output function. For example if we use

$2F(\cdot)$ as the output function instead of $F(\cdot)$, the storage capacity become larger in the case of $2F(\cdot)$ than in that of $F(\cdot)$. On the other hand, the property of the storage capacity on the synapse noise does not depend on the dynamic rage of out function. Thus, we have restricted our discussion to the case that the dynamic range of output function is fixed i.e., $F(\cdot) \leqq 1$. We used some other kinds of nonmonotonic functions [6], and obtained that the results discussed here qualitatively hold as long as the dynamic range of output function is fixed.

4. Conclusion

We have shown the dependence of the storage capacity of the analogue associative memory model with the nonmonotonic neurons, on the static synaptic noise and the static noise in the threshold. We derived its SCSNA order-parameter equations and discussed the influences of the two kinds of noise on the storage capacity. When the monotonicity of neuron is high, the storage capacity decreases rapidly according to the increase of the static synaptic noise. The storage capacity is more sensitive to the increase in the threshold noise level than to the increase in the static synaptic noise level.

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Appendix: Derivation of the SCSNA—The Case with the Static Synaptic Noise and the Threshold Noise

Expressing the internal potential u_i in the equilibrium state in terms of the overlaps (6), we obtain

$$\begin{aligned} u_i &= \sum_{j \neq i}^{N} J_{ij} x_j + \delta_i \\ &= \sum_{\mu=1}^{\alpha N} \xi_i^\mu m_\mu + \sum_{j \neq i}^{N} \delta_{ij} x_j + \delta_i - \alpha x_i. \end{aligned} \tag{A$\cdot$1}$$

The output x_i can be formally expressed as

$$\begin{aligned} x_i &= F\left(\sum_{\mu=1}^{\alpha N} \xi_i^\mu m_\mu + \sum_{j \neq i}^{N} \delta_{ij} x_j + \delta_i - \alpha x_i \right) \\ &= \tilde{F}\left(\sum_{\mu=1}^{\alpha N} \xi_i^\mu m_\mu + \sum_{j \neq i}^{N} \delta_{ij} x_j + \delta_i \right), \end{aligned}$$

where the function $\tilde{F}(\cdot)$ will be determined later. The residual overlap $m_\mu = O(1/\sqrt{N})$, $(\mu \geqq 2)$ is obtained by using the Taylor expansion

$$\begin{aligned} m_\mu &= \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu \tilde{F}\left(\sum_{\nu=1}^{\alpha N} \xi_i^\nu m_\nu + \sum_{j \neq i}^{N} \delta_{ij} x_j + \delta_i \right) \\ &= \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu x_i^{(\mu)} + U m_\mu, \\ &= \frac{1}{N(1-U)} \sum_{i=1}^{N} \xi_i^\mu x_i^{(\mu)} \end{aligned} \tag{A$\cdot$2}$$

where

$$\begin{aligned} x_i^{(\mu)} &= \tilde{F}\left(\sum_{\nu \neq \mu}^{\alpha N} \xi_i^\nu m_\nu + \frac{1}{N} \sum_{\nu \neq \mu}^{\alpha N} \sum_{j \neq i}^{N} \delta_{ij} x_j + \delta_i \right), \\ x_i'^{(\mu)} &= \tilde{F}'\left(\sum_{\nu \neq \mu}^{\alpha N} \xi_i^\nu m_\nu + \frac{1}{N} \sum_{\nu \neq \mu}^{\alpha N} \sum_{j \neq i}^{N} \delta_{ij} x_j + \delta_i \right), \\ U &= \frac{1}{N} \sum_{i=1}^{N} x_i'^{(\mu)}. \end{aligned}$$

Similarly, the second term in the rhs of Eq. (A$\cdot$1) is expressed as

$$\begin{aligned} \sum_{j \neq i}^{N} \delta_{ij} x_j &= \sum_{j \neq i}^{N} \delta_{ij} x_j^{(\delta_{ji})} + \Delta_J^2 U x_i, \\ x_j^{(\delta_{ji})} &= \tilde{F}\left(\sum_{\nu=1}^{\alpha N} \xi_i^\nu m_\nu + \sum_{k \neq i,j}^{N} \delta_{jk} x_k + \delta_i \right). \end{aligned} \tag{A$\cdot$3}$$

Using Eqs. (A$\cdot$2) and (A$\cdot$3), we obtain

$$u_i = \xi_i^1 m_1 + \left[\frac{\alpha U}{1-U} + \Delta_J^2 U\right] x_i + \bar{z},$$

where $\bar{z}$ is the effective noise as follows:

$$\bar{z} = \frac{1}{N(1-U)} \sum_{\mu=2}^{\alpha N} \sum_{j \neq i}^{N} \xi_i^\mu \xi_j^\mu x_j^{(\mu)} + \sum_{j \neq i}^{N} \delta_{ij} x_j^{(\delta_{ji})} + \delta_i.$$

Note that $\bar{z}$ is a summation of uncorrelated random variables, with $< \bar{z} >= 0$ and $< \bar{z} >= \sigma^2$. Thus,

$$\sigma^2 = \alpha r + \Delta_J^2 q + \Delta_h^2,$$

$$q = \frac{1}{N} \sum_{i=1}^{N} (x_i)^2,$$

$$\alpha r = \frac{1}{N(1-U)^2} \sum_{\mu=2}^{\alpha N} \sum_{j \neq i}^{N} (\xi_i^\mu)^2 (\xi_j^\mu)^2 (x_j^{(\mu)})^2 = \frac{\alpha q}{(1-U)^2}.$$

Replacing $u_i \to u$, $\xi_i^1 m_1 \to m$, and $\tilde{F}(u) \to Y$, and setting $z = \bar{z}/\sigma$, Eqs. (8)–(14) are obtained, since we have discussed the odd function $F(\cdot)$.

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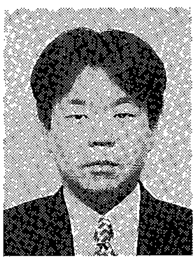

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